

On Milnor Algebras of Invertible Polynomials

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ABSTRACT. We construct explicit monomial bases in Milnor algebras of invertible polynomials and calculate the signature of Gorenstein quadratic form. As an application we obtain exact upper estimates for the gradient degree of invertible polynomials and show that they improve the estimates by Petrovsky numbers. © 2019 Bull. Georg. Natl. Acad. Sci.

Key words: quasihomogeneous polynomial, invertible polynomial, gradient mapping, multiplicity, moduli algebra, Mather-Yau theorem, Milnor algebra, monomial basis, Gorenstein quadratic form, signature, mapping degree, Petrovsky number

We begin with some definitions and recollections needed for formulation of the main results. Let us denote by $\mathbb{R}[x_1, \dots, x_n]$ ($\mathbb{R}\{x_1, \dots, x_n\}$) the algebra of polynomials (formal power series) in n indeterminates with real coefficients and by $M[n]$ ($M\{n\}$) the maximal ideal of polynomials (formal power series) with zero free term. For a polynomial $f \in M[n]$, denote by $\partial f = \text{grad } f$ its gradient. If ∂f vanishes at the origin then we denote by $(f, \partial f)$ the ideal in $\mathbb{R}\{n\}$ generated by f and its partial derivatives. The *moduli algebra* of f is defined as the factor-algebra $A(f) = \mathbb{R}\{n\}/(f, \partial f)$ (see, e.g., [1]). If f has an isolated critical point at the origin of $\mathbb{C}^n$ then $A(f)$ is a finite dimensional $\mathbb{R}$ -algebra and its dimension is called the *multiplicity* of f at the origin. By the famous Mather-Yau theorem moduli algebra is a complete invariant of the germ of f at the origin. More precisely, if two function germs depending on the same number of variables have isomorphic moduli algebras then they can be transformed into each other by a biholomorphism (see, e.g., [2, 3]).

The algebraic structure of $A(f)$ is also important in many applications of singularity theory [4, 5]. With a view toward some issues studied below we point out that there is a natural non-degenerate quadratic form $Q(f)$ on $A(f)$ such that the Euler characteristic of local level surfaces of f can be expressed using the signature $\text{sig } Q(f)$ of $Q(f)$ [5]. The latter signature in turn can be expressed through the multiplicity and the dimension of maximal ideal with zero square in $A(f)$ [6, 7]. These results will be essentially used below.

The collection of exponents of monomials present in polynomial f is called the support $S(f)$ of f . The convex hull of support in $\mathbb{Z}^n$ is called the Newton polyhedron $N(f)$ of f . Polynomial $f \in M[n]$ is called *quasihomogeneous* if there exist positive rational numbers (w_j, d) such that, for each monomial $e \in S(f)$, the

w -degree of e equals d . Each such collection (w, d) is called *qh-type* of f and d is called the *w-degree* of f . If f is quasihomogeneous then by Euler's theorem $f \in (\partial f)$ so its moduli algebra is isomorphic to the so-called *Milnor algebra* $R\{n\}/(\partial f)$.

In such a case, the multiplicity is computed as $M = \prod_{i=1}^n \frac{(1-w_i)}{w_i}$.

A polynomial $f \in M[n]$ is called an *invertible polynomial* [8, 9] if it satisfies three conditions:

1. f has an isolated critical point at the origin;
2. f has a unique qh type with $d = 1$;
3. The number of monomials in $S(f)$ equals n .

Such polynomials appear in a number of problems of modern mathematical physics and some properties of their Milnor algebras can be interpreted in terms of the so-called *mirror symmetry* [8, 9]. An important result proven in [9] states that up to a change of coordinates each invertible polynomial can be decomposed as a direct sum of polynomials of three types:

- I. $x_1^{k_1} + x_2^{k_2} + \dots + x_n^{k_n}$ (Fermat)
- II. $x_1^{k_1} x_2 + x_2^{k_2} + \dots + x_n^{k_n}$ (chain)
- III. $x_1^{k_1} x_2 + x_2^{k_2} + \dots + x_n^{k_n} x_1$ (loop)

It turns out that in many issues concerned with invertible polynomials it is sufficient to consider only polynomials of these three basic types. In particular, this refers to the structure of their Milnor algebras which is our main concern in the sequel. Some of our results are only proven for *homogeneous* invertible potentials.

The main aims of this paper can now be formulated as follows:

1. compute the qh-type and Poincare polynomial of an invertible polynomial;
2. compute a monomial basis of Milnor algebra and the signature of Gorenstein quadratic form;
3. estimate the maximal possible value of signature for given n and compare it with the known estimates of gradient degree.

In the sequel we compute the *qh-type* and multiplicity of invertible polynomials. This enables us to construct an explicit monomial basis and calculate the signature of Gorenstein quadratic form in terms of *qh-type* in some cases. As a corollary we obtain upper estimates for the Gorenstein signatures and compare them with the general estimates in terms of Petrovsky numbers given in [5].

We proceed by further remarks on qh-polynomials. Notice that in each of three normal forms the input is a collection of n natural numbers $k=(k_j)$. Solving a linear system for weights w_j and using an explicit formula for multiplicity of qh polynomial the qh-type and multiplicity can be explicitly expressed through vector k in each of three cases as (see, e.g., [1]). It is well-known that the qh-weights of elements of each monomial basis are the same and the number of monomials of each weight can be obtained from the formula for Poincare polynomial P_f of Milnor algebra of *qh* polynomial of *qh-type* (w, d) given in [1].

It is also known that the highest qh weight of basis monomials is the qh degree of hessian $h(f) = \det(\partial^2 f / \partial x_i \partial x_j)$. Moreover, the principal ideal (h_f) generated by h_f in $A(f)$ is the *socle*, i.e. the unique minimal ideal of $A(f)$. Information about the structure of Milnor algebras can be explicated in the case of homogeneous invertible potential of degree d . Then the multiplicity equals $(d-1)^n$ and algebraic degree of hessian equals $n(d-2)$. The Poincare polynomial is just $(t^d - 1)^n / (t-1)^n$. Moreover, for $k < d-1$ the number of basic monomials of degree k in $A(f)$ equals $(k+n-1)!/k!(n-1)!$, and the numbers of monomials of other degrees are also given by explicit formulae which can be found in [1].

Using this information and explicit formulae for partial derivatives of f it appears possible to construct monomial bases in Milnor algebra in several special cases. In this paper we present results referring to arbitrary invertible potentials with $n=2$ and $n=3$. We also present analogous results for homogeneous invertible potentials with arbitrary number of variables.

For $n=2$, in each of the three irreducible cases the weights and monomial bases can be explicitly computed and have the following form:

I. weights: $\left(\frac{1}{a}, \frac{1}{b}\right)$, multiplicity $M = \frac{1-w_1}{w_1} \frac{1-w_2}{w_2} = (a-1)(b-1)$, monomial basis:

$$\{x^m y^n \mid m \leq (a-2), n \leq (b-2)\};$$

II. weights: $\left(\frac{b-1}{ab-1}, \frac{a-1}{ab-1}\right)$, multiplicity $M = \frac{1-w_1}{w_1} \frac{1-w_2}{w_2} = ab$, monomial basis:

$$\{x^m y^n \mid m \leq (a-1), n \leq (b-1)\};$$

III. weights $\left(\frac{b-1}{ab}, \frac{1}{b}\right)$, multiplicity $M = \frac{1-w_1}{w_1} \frac{1-w_2}{w_2} = (a-1)b+1$, monomial basis:

$$\{x^m y^n \mid m \leq (a-2), n \leq (b-1)\} + \{x^a\}.$$

For $n=3$, in each of the three irreducible cases the weights and the monomial bases are as follows:

I. weights $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, multiplicity $M = \frac{1-w_1}{w_1} \frac{1-w_2}{w_2} \frac{1-w_3}{w_3} = (a-1)(b-1)(c-1)$, monomial basis:

$$\{x^m y^n z^k \mid m \leq (a-2), n \leq (b-2), k \leq (c-2)\};$$

II. weights $\left(\frac{1}{a}, \frac{1}{b}, \frac{b-1}{bc}\right)$, $M = \frac{1-w_1}{w_1} \frac{1-w_2}{w_2} \frac{1-w_3}{w_3} = (a-1)(bc-b+1)(c-1)$, basis:

$$\{x^m y^n z^k \mid m \leq (a-2), n \leq (b-1), k \leq (c-2)\} \cup \{z^{c-1} x^m \mid m \leq a-2\};$$

III. weights $\left(\frac{1}{a}, \frac{c-1}{bc-1}, \frac{b-1}{bc-1}\right)$, multiplicity $M = \frac{1-w_1}{w_1} \frac{1-w_2}{w_2} \frac{1-w_3}{w_3} = (a-1)bc$, monomial basis:

$$\{x^m y^n z^k \mid m \leq (a-2), n \leq (b-1), k \leq (c-1)\}.$$

To estimate the signature of Gorenstein quadratic form recall that according to [2, 5] $\text{sig } Q_f = m - 2 \dim I$, where I is any ideal with $I^2 = 0$ which is maximal among the ideals with this property. The reciprocity of Poincare polynomial implies that $\dim I$ does not exceed the number of basis monomials of w -degree equal to one half of the maximal degree equal to $w\text{-deg } h_f$. To find the exact value of $\dim I$ it is necessary to compute the products of monomials of w -degree equal to $1/2 w\text{-deg } h_f$. The main result of [6, 7] states that this signature is equal to the absolute value of gradient degree of polynomial f . Below we use these general results to obtain some information in the case of invertible potentials.

Example. As a simple illustration we consider the polynomial $f = x^2 y + xy^2$. Here we can put $w_1 = w_2 = 1$, $d = 3$ so that $m = \dim A(f) = 4$. Indeed, $(\partial f) = (2xy + y^2, x^2 + 2xy)$ and it is obvious that a monomial basis in $A(f)$ is given by classes of monomials $1, x, y, xy$. Further, hessian $h(f) = 4xy$ and the socle is (xy) . Obviously, in $A(f)$ we have identities $x^2 = y^2 = -1/2 xy \pmod{(\partial f)}$. Thus, the socle (xy) is a maximal ideal with zero square. So $\dim I = 1$ and $|\deg \text{grad } f| = 4 - 2 = 2$, which is indeed so as can be directly verified (actually, $\deg \text{grad } f = -2$).

Analogous computations can be done algorithmically for arbitrary invertible polynomial but this requires considerable amount of computations. However, computation of $\dim I$ is far from simple. We were only able to effectively compute $\dim I$ only for invertible potentials in two and three variables.

These results enable us to give exact estimates for the gradient degree of invertible polynomials using the main result of [7]. The known results on roots of sparse polynomials suggest that the gradient mappings of invertible polynomials should have much less real zeroes than the corresponding Bezout number [5]. Hence the gradient degree should be also be rather small as compared with the algebraic degree. It is known that an upper bound for the absolute gradient degree of a sparse homogeneous polynomial P of degree d in n variables is given by expression which depends only on n and number of monomials in [5]. In other words, it does not depend on the degree of polynomial. It follows that the absolute gradient degree of invertible polynomial does not exceed an estimate depending only on the number of variables. The results presented above enable us to prove an explicit version of this conjecture in cases where the number of variables does not exceed four.

Theorem 1. For $n \leq 4$, the absolute value of gradient degree of any invertible polynomial in n variables does not exceed n .

This follows directly from the above results on monomial bases and dimension of maximal ideal with zero square of invertible polynomial.

Since the gradient degree of direct sum of functions is equal to the sum of gradient degrees we conclude that $\dim I$ can be estimated for any invertible polynomial if its decomposition in direct sum of basic three types is known. Taking into account Theorem 1 this suggests the following conjecture.

Conjecture 1. For any invertible polynomial in n variables, the absolute value of $\deg \text{grad } f$ does not exceed n .

This conjecture yields a sharper estimate than the general estimate of gradient degree by in terms of Petrovsky number defined as follows (see, e.g., [5]). Let F be an n -ary d -form (homogeneous polynomial of degree d in n variables) with an isolated critical point at the origin. Denote by $dg F = \deg_0 \text{grad } F$ the local degree of gradient mapping. The Petrovsky number $P(n, d)$ is defined as follows. If both n and d are odd then $P(n, d) = 0$. Otherwise, $P(n, d)$ is the number of integer points in the first orthant of $\mathbb{R}^n$ such that the sum of their coordinates equals $n(d-2)/2$. If $P(n, d)$ does not vanish it can be estimated by $\frac{(p+n-1)!}{p!(n-1)!}$, where $p = n(d-2)/2$.

Using this notation the main result of [5] can be formulated as follows. For any n -ary d -form F with an isolated critical point at the origin, the following inequality holds true:

$$|dg F| \leq P(n, d).$$

As was shown in [5] this estimate is exact, in other words, there exist n -ary d -forms with the absolute value of gradient degree equal to the Petrovsky number. Comparing this inequality with Theorem 1 and Conjecture 1 one concludes that, for invertible polynomials, our estimate is much sharper than the above estimate by Petrovsky number.

Theorem 1 can also be used to explicate results on the Euler characteristic of local level surfaces of homogeneous polynomials established in [7]. To this end we need further definitions and notation. Let F be an n -ary d -form (homogeneous polynomial of degree d in n variables) with an isolated critical point at the origin. Denote again by $dg F = \deg_0 \text{grad } F$ the local degree of gradient mapping at the origin and by $X(a, r)$ the intersection of level surface $\{F = a\}$ with the ball B_r of radius r centered at the origin. With these assumptions, it was shown in [7] that, for sufficiently small a , $r > 0$, the local Euler characteristics can be computed by the formula

$$\chi(X(a,r)) = (-1)^n - dg F,$$

where $\chi(X(a,r))$ denotes the Euler characteristic of local level surface.

Theorem 2. For $n=2, 3$, the absolute value of the Euler characteristic of local level surface of invertible polynomial does not exceed 3 and 4 respectively.

Analogously, if our Conjecture 1 holds true then $\chi(X(a,r))$ of an invertible polynomial is much smaller than for arbitrary n -ary d -forms.

Conjecture 2. The absolute value of the Euler characteristic of local level surfaces of an invertible polynomial does not exceed $n - 1$.

This means that the topology of level surfaces of homogeneous invertible polynomials is much simpler than of general homogeneous hypersurfaces of the same algebraic degree.

მათემატიკა

შებრუნებადი მრავალწევრების მილნორის ალგებრების შესახებ

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აგებულია მონომიალური ბაზისი შებრუნებადი მრავალწევრის მილნორის ალგებრაში და გამოთვლილია გორენშტაინის კვადრატული ფორმის სიგნატურა. ამ შედეგების დახმარებით მიღებულია შებრუნებადი მრავალწევრის გრადიენტული ხარისხის შეფასება, რომელიც უფრო ზუსტია ვიდრე ცნობილი შეფასება პეტროვსკის რიცხვის ტერმინებში.

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