

A Modified Voronoi Diagram in the R^n Space and Modules of the Application Software Package

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(Presented by Academy Member Ramaz Khurodze)

Abstract. The classical method of the Voronoi diagram implies the homogeneity of objects, which is often far from reality and consequently limits not only the scope of its application, but also its capabilities. When adding potential power to objects, the modified Voronoi method gains additional function. This paper presents a method for constructing a modified Voronoi diagram in R^n space and mathematical modules of the application software system. The technical architecture and implementation of the website created on the basis of the obtained algorithms represent a technically well-implemented platform using modern web technologies for visualization and data analysis. Its modular architecture and correct code ensure ease of further development and modification. © 2025 Bull. Natl. Acad. Sci. Georg.

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The classical Voronoi diagram method assumes the homogeneity of objects (Akhobadze et al., 2024), which, of course, is far from reality and accordingly limits not only the scope of its application, but also its capabilities. Giving objects potential power, the modified Voronoi method additionally acquires another function (Akhobadze & Kurtskhalia, 2022).

The works (Zakharkin, 2016; Akhobadze & Kvitsinashvili, 2023; Abralava et al., 2024) consider examples of using the modernized Voronoi diagram to provide comfortable housing for the population and a safe environment for tourists; to determine safe routes in computer games – as an AI modules; for optimal planning of economic processes, etc.

All these problems belong to the class of spatial relation problems (spatial relations are any information characterizing the connection of objects in space).

The article describes a method for constructing a modified Voronoi diagram in R^n space and the principles of constructing an applied software package. The following theorem is true:

Theorem: If the distance between two points A and B in space R^n is equal to l and $k > 1$ is any number, then the set of all points in space R^n that are k – times farther from the point A than from the point B is a n – dimensional sphere with radius $\frac{kl}{k^2 - 1}$ and center located on the $(A; B)$ ray at a distance $\frac{k^2 l}{k^2 - 1}$ from the point A .

In the paper (Akhobadze & Kurtskhalia, 2022), a proof of this theorem is given for the case $n = 2$ (the theorem is proved analogously for any natural number n).

Analytical definition of the object potential: Let $\{S_1; S_2; \dots; S_N\}$ be the set of objects of the region (urban fabric) under consideration, and $\{P_1; P_2; \dots; P_M\}$ be the set of all possible characteristic parameters of the objects under consideration. Not all parameters are equally important for a given object. Let us denote by m_{ir} – the weight of the parameter P_r for the i -th object $\left(\sum_{r=1}^M m_{ir} = 1\right)$ and by V_{ir} – the value showing the volume of the i -th object in relation to the parameter P_r (for example, V_{11} shows the volume of electricity consumed by object S_1 , V_{12} shows the volume of gas consumed by object S_1 , etc.).

Let us denote by λ_{ir} the element at the intersection of the i -th row and the r -th column of the incidence matrix, corresponding to the relationship between the set of objects $\{S_1; S_2; \dots; S_N\}$ of the region (urban fabric) under consideration and by all possible characteristic parameters $\{P_1; P_2; \dots; P_M\}$

$$\lambda_{ir} = \begin{cases} 1, & \text{if } P_r \text{ is a characteristic of object } S_i \\ 0, & \text{if } P_r \text{ is not a characteristic of object } S_i. \end{cases} \quad (1)$$

The potential of the i -th object U_i is a vector quantity: $U_i(u_{i1}; u_{i2}; \dots; u_{iM})$, the numerical value of each coordinate of which is calculated using the formula:

$$u_{ir} = \lambda_{ir} \cdot m_{ir} \cdot V_{ir}.$$

The numerical value (modulus) of the full potential of an object is determined by the coordinates of the specified vector, as the length of the vector: $|U_i| = \sqrt{\sum_{r=1}^M (u_{ir})^2}$.

The prerequisite for mathematical modeling - planning of n -dimensional space can be:

The value of the boundaries (dimensions) of the entire design area; all objects that should be located on a given territory are known; the potential of all these objects is known.

The size (potential) of the entire designed territory is known and its development is carried out gradually.

Below, we will consider the algorithms that are implemented in the application software package.

We enter a map of the designed territory into the software system.

Case I:

The location of objects on the design territory is known; that is, the coordinates of each i -th object marked on the map of the design territory $S_i(x_1^{(i)}; x_2^{(i)}; \dots; x_n^{(i)})$ are known. Roads are marked, the road capacity is indicated.

The characteristic parameters of the designed objects are given $\{P_1; P_2; \dots; P_M\}$

The values of m_{ir} and V_{ir} are known for each i and each r .

Given a matrix λ_{ir} ;

Let E_{ij} denote the set of all points in E where the influence of point S_i is greater than the influence of point S_j . To determine the set E_{ij} , we need to perform several calculations: for each i and j , we calculate the distance between the i -th and j -th objects

$$S_{ij} = \sqrt{(x_1^{(i)} - x_1^{(j)})^2 + (x_2^{(i)} - x_2^{(j)})^2 + \dots + (x_n^{(i)} - x_n^{(j)})^2},$$

and the potentials of the i -th and j -th $|U_i|$ and $|U_j| \cdot k_{ij} = \frac{|U_i|}{|U_j|}$.

It follows from the above theorem that if $k_{ij} > 1$, then the set of points at which the influence of the point (object) $S_i(x_1^{(i)}; x_2^{(i)}; \dots; x_n^{(i)})$ is greater than the influence of the point (object) $S_j(x_1^{(j)}; x_2^{(j)}; \dots; x_n^{(j)})$ is the set of points located outside the sphere specified in the theorem; And the set of points at which the influence of the point (object) $S_j(x_1^{(j)}; x_2^{(j)}; \dots; x_n^{(j)})$ is greater than the influence of the point (object) $S_i(x_1^{(i)}; x_2^{(i)}; \dots; x_n^{(i)})$ are the points located inside the sphere specified in the theorem.

If $k_{ij} = 1$, then the set of points in which the influence of the point (object) $S_i(x_1^{(i)}; x_2^{(i)}; \dots; x_n^{(i)})$ is greater than the influence of the point (object) $S_j(x_1^{(j)}; x_2^{(j)}; \dots; x_n^{(j)})$, is the half-spaces obtained by the median of the segment connecting the points $S_i(x_1^{(i)}; x_2^{(i)}; \dots; x_n^{(i)})$ and $S_j(x_1^{(j)}; x_2^{(j)}; \dots; x_n^{(j)})$, in which the point $S_i(x_1^{(i)}; x_2^{(i)}; \dots; x_n^{(i)})$ is located.

Let us call the Voronoi cell ("coverage zone") of the point S_i the set $E_i = \bigcap_{\substack{j=1 \\ j \neq i}}^N E_{ij}$. It is clear that $E = \bigcup_{i=1}^N E_i$ – the region E under consideration is divided into Voronoi cells of points (objects) $S_i(x_1^{(i)}; x_2^{(i)}; \dots; x_n^{(i)})$, $i \in \overline{1; N}$ – „coverage zones“ of points $S_i(x_1^{(i)}; x_2^{(i)}; \dots; x_n^{(i)})$, $i \in \overline{1; N}$.

II. Case. Design (placement) of objects on the design area is carried out gradually.

Let there be free space on the design area and the potential of the design area allows placing new objects $(O_N^{(1)}; O_N^{(2)}; \dots; O_N^{(m)})$ on the area under consideration, in time intervals $[t_1; t_1 + \Delta t_1]; [t_2; t_2 + \Delta t_2]; \dots; [t_n; t_n + \Delta t_n]$ (without loss of generality we can assume that $t_1 < t_2 < \dots < t_m$. the potentials of which are respectively $|U_N^{(1)}| + |U_N^{(2)}| + \dots + |U_N^{(m)}|$. Let us assume that new objects are to be placed at points

$$(x_{N_1}^{(1)}; x_{N_2}^{(1)}; \dots; x_{N_n}^{(1)}); (x_{N_1}^{(2)}; x_{N_2}^{(2)}; \dots; x_{N_n}^{(2)}); \dots; (x_{N_1}^{(m)}; x_{N_2}^{(m)}; \dots; x_{N_n}^{(m)}).$$

Let us denote the potential of free space by $|U_0|$. Let us require that the condition be satisfied: $|U_N^{(1)}| + |U_N^{(2)}| + \dots + |U_N^{(m)}| \leq |U_0|$.

Let's construct Voronoi cells for objects $(O_N^{(1)}; O_N^{(2)}; \dots; O_N^{(m)})$ iteratively. First, let's construct a Voronoi cell for object $O_N^{(1)}$.

If the volume of free space is S_0 , then logically the volume of the region covered by object $O_N^{(1)}$ should be $S_1 = S_0 \cdot |U_N^{(1)}| / |U_0|$.

Let's say $O_0^{(1)}$ is an object whose potential is $|U_o| - |U_N^{(1)}|$. Our goal is to find a point where object $O_0^{(1)}$ should be placed, so that if we divide the free space proportionally to the potentials of objects $O_0^{(1)}$ and $O_N^{(1)}$, then the volume of the region covered by object $O_N^{(1)}$ is S_1 .

First, we define the distance between these objects, a .

$$\text{Let } k_1 = \frac{|U_o| - |U_N^{(1)}|}{|U_N^{(1)}|}.$$

As shown above, the radius of a sphere is calculated by the formula $R = \frac{k_1 a}{k_1^2 - 1}$, so we must choose a such that the volume of a core of radius R is equal to the size of S_1 :

$$C_n \left(\frac{k_1 a}{k_1^2 - 1} \right)^n = S_1. \quad (1)$$

In (1), C_n is calculated by the formula:

$$C_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)},$$

where $\Gamma(x)$ is the gamma function. From (1) we obtain that

$$a = \frac{k_1^2 - 1}{k_1} \cdot \sqrt[n]{\frac{S_1}{C_n}}. \quad (2)$$

Therefore, the object $O_0^{(1)}$ must be placed at a point in free space located on a sphere with center $(x_{N_1}^{(1)}; x_{N_2}^{(1)}; \dots; x_{N_n}^{(1)})$ and radius a , where a is calculated using formula (2).

Let us determine where the center O of the coverage zone of point $O_0^{(1)}$ can be located $O_N^{(1)}(x_{N_1}^{(1)}; x_{N_2}^{(1)}; \dots; x_{N_n}^{(1)})$. As we know, point O must be at distance $\frac{a}{k_1^2 - 1}$ from point $O_N^{(1)}(x_{N_1}^{(1)}; x_{N_2}^{(1)}; \dots; x_{N_n}^{(1)})$. Therefore, point O must be a point in free space located on a sphere with radius $\frac{a}{k_1^2 - 1}$ and center at $(x_{N_1}^{(1)}; x_{N_2}^{(1)}; \dots; x_{N_n}^{(1)})$, while the circle with center at O and radius $\frac{k_1 a}{k_1^2 - 1}$ must be a subset of free space.

Suppose that $2 \leq i \leq n-1$ and we have already constructed the Voronoi cells of objects $(O_N^{(1)}; O_N^{(2)}; \dots; O_N^{(i-1)})$ the volumes of which are equal to $S_1; S_2; \dots; S_{i-1}$. respectively. Let us construct the Voronoi cell of the $O_N^{(i)}$ object using the same principle by which we constructed the Voronoi cell of the $O_N^{(1)}$ object. It is logical that the volume of the Voronoi cell S_i , of the object $O_N^{(i)}$ should be equal to the value:

$$S_i = (S_0 - (S_1 + S_2 + \dots + S_{i-1})) \cdot |U_N^{(i)}| / \left(|U_o| - (|U_N^{(1)}| + |U_N^{(2)}| + \dots + |U_N^{(i-1)}|) \right).$$

Let's assume that $O_0^{(i)}$ is an object with potential $|U_o| - (|U_N^{(1)}| + |U_N^{(2)}| + \dots + |U_N^{(i-1)}|)$. As described above (when constructing the Voronoi cell of the object $O_N^{(i)}$, we find the point at which the object $O_0^{(i)}$ should be placed.

If we divide the free space available at the moment proportionally to the potentials of objects $O_0^{(i)}$ and $O_N^{(i)}$, then the volume of the Voronoi cell of the object $O_N^{(i)}$ will be equal to S_1 .

The technical architecture and implementation of the website, which ensures the implementation of the above methodology and algorithms, is technically well-implemented platform that uses modern web technologies for data visualization and analysis. Its modular architecture and correct code ensure its further development and modification.

ინფორმატიკა

ვორონოის მოდიფიცირებული დიაგრამის აგების ალგორითმი R^n სივრცეში და შესაბამისი გამოყენებითი პროგრამული პაკეტის მათემატიკური ინსტრუმენტები

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(წარმოდგენილია აკადემიის წევრის რ. ხუროძის მიერ)

ვორონოის დიაგრამის კლასიკური მეთოდი გულისხმობს ობიექტების ერთგვაროვნებას, რაც, ცხადია, შორსაა რეალობისაგან და, შესაბამისად, ზღუდავს არა მარტო მისი გამოყენების არეალს, არამედ შესაძლებლობებსაც. ობიექტთა შესახებ პოტენციური სიმძლავრის დამატებისას, ვორონოის მოდიფიცირებული მეთოდი, დამატებით იძენს სხვა ფუნქციასაც. ნაშრომში წარმოდგენილია მოდიფიცირებული ვორონოის დიაგრამის აგების მეთოდი R^n სივრცეში და გამოყენებითი პროგრამული პაკეტის მოდულები. მიღებული ალგორითმების ბაზაზე შექმნილი ვებ-გვერდის ტექნიკური არქიტექტურა და განხორციელება წარმოადგენს ტექნიკურად კარგად დანერგილ პლატფორმას, რომელიც იყენებს თანამედროვე ვებ-ტექნოლოგიებს, მონაცემთა ვიზუალიზაციისა და ანალიზისთვის. მისი მოდულური არქიტექტურა და სწორი კოდი უზრუნველყოფს მის შემდგომ განვითარებასა და მოდიფიკაციას.

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